This document will not simulate the actual final, but it is meant moreso as a comprehensive review!

1. Your company is engineering a new type of airplane wing for manufacturing that is being constructed according to the curve $y = e^x \sin(x)$ on the interval $[0, \pi]$. As a member of the engineering team, your job is to determine the amount of material (in square meters) that is needed to cover the surface of the wing.

$A = \int_0^\pi e^x \sin x \, dx$ → Int by parts using LIATE.

$u = \sin x$, $dv = e^x dx$
$du = \cos x \, dx$, $v = e^x$

$= e^x \sin x \Big|_0^\pi - \int_0^\pi e^x \cos x \, dx$

$A = -\int_0^\pi e^x \cos x \, dx$

$u = \cos x$, $dv = e^x dx$
$du = -\sin x \, dx$, $v = e^x$

$= -\left[e^x \cos x \Big|_0^\pi + \int_0^\pi e^x \sin x \, dx\right]$

$\int_0^\pi e^x \sin x\,dx = -\left[-e^\pi - 1 + \int_0^\pi e^x \sin x \, dx\right]$

→ $2\int_0^\pi e^x \sin x \, dx = e^\pi + 1$

$\boxed{\int_0^\pi e^x \sin x \, dx = \frac{e^\pi + 1}{2}}$

2. A physicist comes to you and asks for your help as a Calc 2 student! He found that the area that particles cover over a certain time period behave according to the equation

$$A(t) = 2\int \frac{\sin(t)\sec^5(t) - \sin(t)\sec^3(t)}{\sin(2t)\cot(t)\csc^2(t) \,\cancel{+}\, - \sin(2t)\cot(t)} dt$$

TYPO

He can plug this into *Mathematica* and get an answer, but unfortunately he needs to know a few intermediate steps too for data purposes. Determine the time dependent area equation that the physicist needs, showing all your steps (and not leaving it to a computer)!

$A(t) = \int \frac{2\sin t \sec^3 t(\sec^2 t - 1)}{\sin(2t)\cot(t)(\csc^2 t - 1)} dt$

$= \int \frac{\cancel{2\sin t} \sec^3 t \tan^2 t}{\cancel{2\sin t} \cos t \cot t \cot^2 t} dt$

$= \int \sec^4 t \tan^5 t \, dt$

$= \int \sec^2 t \sec^2 t \tan^5 t \, dt$

$= \int \sec^2 t(1 + \tan^2 t)\tan^5 t$

$u = \tan(t)$
$du = \sec^2 t \, dt$

$= \int (u^5 + u^7) dt$

$= \frac{u^6}{6} + \frac{u^8}{8} + C = \boxed{\frac{\tan^6 t}{6} + \frac{\tan^8 t}{8} + C}$

3. Determine the area under the curve, knowing that $x^4 - 6x^3 + 10x^2 - 6x + 9 = (x^2 + 1)g(x)$

$g(x) = x^2 - 6x + 9 = (x-3)^2$ (by long division)

$$y = \frac{7x^3 - 20x^2 + 5x + 16}{x^4 - 6x^3 + 10x^2 - 6x + 9}$$

partial fractions $= \frac{2x+3}{x^2+1} + \frac{5}{x-3} + \frac{4}{(x-3)^2}$

(a) on the interval $[4, 10]$

$= \int_4^{10} \frac{2x\,dx}{x^2+1} + \int_4^{10} \frac{3}{x^2+1} dx + \int_4^{10} \frac{5}{(x-3)} dx + \int_4^{10} \frac{4}{(x-3)^2} dx$

$= \boxed{\ln\left(\frac{101}{17}\right) + 3\arctan 10 - 3\arctan 4 + 5\ln 7 - \frac{4}{7} + 4}$

(b) on the interval $[4, \infty]$

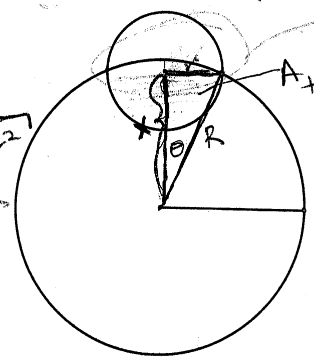
This will be divergent in the first integral above $\Rightarrow \int_4^\infty \frac{2x}{x^2+1} dx = \infty$, so the rest will result in a divergent integral as well!

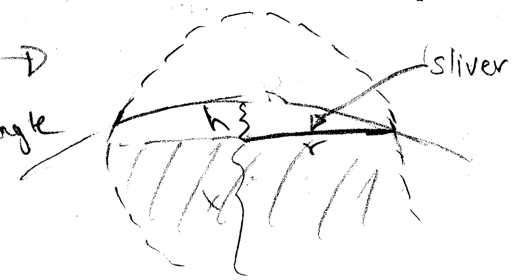
4. Consider the following picture of two circles, the smaller one of radius r and the larger of radius R. Determine the area that is inside the smaller circle, but outside of the larger circle. (This crescent shaped region is called a *lune*).

$R^2 = r^2 + x^2$; $R = x + h$

$A_{lune} = \frac{1}{2} A_{small\,circle} + A_{sliver}$

$= \frac{1}{2}\pi r^2 + \frac{R^2}{2}\sin^{-1}\left(\frac{r}{R}\right) - r\sqrt{R^2 - r^2}$





$A_{sliver} = A_{sector} - 2A_{triangle}$

$= \frac{R^2}{2}\theta - 2\frac{1}{2} r \cdot x$

$= R^2 \sin^{-1}\left(\frac{r}{R}\right) - r\sqrt{R^2 - r^2}$

5. Find R_4, L_4, M_4, and T_4 for $\int_4^6 \ln(x^3+2)\,dx$. Label each as an over or underestimate.

$L_4 \approx 9.3469$ underestimate (increasing)

$R_4 \approx 9.9944$ overestimate (increasing)

$M_4 \approx 9.6529$ overestimate (concave down)

$T_4 \approx 9.6456$ underestimate (concave down)

6. State why the integral cannot be evaluated in its current form. Find a method to evaluate it and conclude either convergence (and stating its value) or divergence.

$$\int_4^{12} \frac{1}{\sqrt{x^2 - 16x + 48}}\,dx$$

It does not exist at EITHER bound

Using "lazy" notation for now,

$= \int_4^{12} \frac{dx}{\sqrt{x^2 - 16 + 64 - 16}}$

$= \int_4^{12} \frac{dx}{\sqrt{(x-8)^2 - 16}}$

$u = x - 8$, $du = dx$

$= \int_{-4}^{4} \frac{du}{\sqrt{u^2 - 16}}$

Let $u = 4\sec\theta$, $du = 4\sec\theta\tan\theta\,d\theta$

$\Rightarrow \theta = \sec^{-1}\left(\frac{u}{4}\right)$

$= \int_{\pi}^{2\pi} \frac{4\sec\theta\tan\theta\,d\theta}{4\sqrt{\sec^2\theta - 1}}$

$= \int_{\pi}^{2\pi} \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta|\Big|_{\pi}^{2\pi}$

$= \lim_{\substack{t\to 2\pi^- \\ n\to\pi^+}} \ln|\sec\theta + \tan\theta|\Big|_n^t$

$= \lim_{\substack{t\to 2\pi^- \\ n\to\pi^+}} \ln\left|\frac{\sec t + \tan t}{\sec n + \tan n}\right|$

$\approx \ln\left|\frac{1 + "\infty"}{-1 + "-\infty"}\right|$

cannot conclude convergence, so the integral is divergent

7. A local garage is creating a new gasket for an engine. After doing some modeling of the gasket, they find that the function $f(x) = \frac{1}{6}x^3 + \frac{1}{2x}$ from $x = \frac{1}{2}$ cm to $x = 3$ cm best models the edge of the gasket.

$f'(x) = \frac{1}{2}x^2 - \frac{1}{2x^2}$

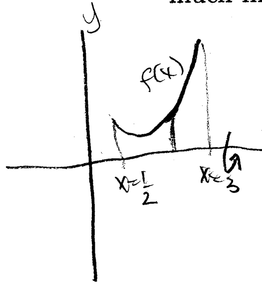
(a) How much material is required to create the edge of the gasket? ⇝ arc length!

$$A = \int_{\frac{1}{2}}^{3} \sqrt{1 + [f'(x)]^2}\,dx = \int_{\frac{1}{2}}^{3} \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}}$$

$$= \int_{\frac{1}{2}}^{3} \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}\,dx = \int_{\frac{1}{2}}^{3} \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)dx$$

$$= \frac{1}{6}x^3 - \frac{1}{2}x^{-1}\Big|_{\frac{1}{2}}^{3} = \boxed{\frac{85}{16}\text{ cm}}$$

(b) As they begin production, the rest of the gasket is made by revolving it around the ~~x~~ y-axis. How much material is required to produce one gasket?

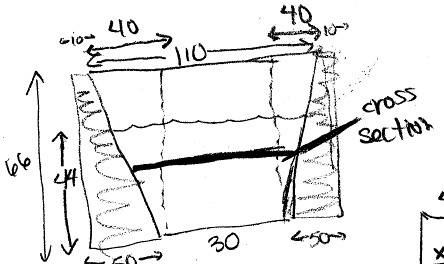


$$G = 2\pi \int_{\frac{1}{2}}^{3} x \sqrt{1 + [f'(x)]^2}\,dx$$

$$= 2\pi \int_{\frac{1}{2}}^{3} x \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)dx$$

$$= 2\pi \int_{\frac{1}{2}}^{3} \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-1}\right)dx \approx \boxed{69.147\text{ cm}^3}$$

8. The Panama Canal, approximately 48 miles in length, has a series of locks that ships must pass through in order to get through the canal. The water in the lock is roughly in the shape of an isosceles trapezoid with its width at the top being 110 feet wide (not including 10 feet on each side for the thickness of each wall) and its width at the bottom of the lock being 30 feet wide (as each wall is 50 feet thick). Knowing that the height of the lock is 66 feet, find the hydrostatic force on the lock when it is 2/3 full of water.



$A_{cross} = (\underbrace{30 + 2x}_{length})\underbrace{dy}_{width}$

$= \left(30 + \frac{40}{33}y\right)dy$

$\frac{x}{40} = \frac{y}{66}$

$x = \frac{20}{33}y$

$$F = \rho \int_{Bottom}^{Top} A_{cross}\left(\text{Height}_{water} - y\right)$$

$$= (62.5)\int_0^{44} \left(30 + \frac{40}{33}y\right)(44 - y)\,dy$$

$$= 62.5 \int_0^{44} \left(-\frac{40}{33}y^2 + \frac{70}{3}y + 1320\right)dy$$

$$\approx \boxed{2{,}890{,}555.6\text{ ft lbs}}$$

9. Consider the functions defined by

$$f(x) = \begin{cases} -3x, & \text{if } -2 \le x < 0 \\ x^4, & \text{if } 0 \le x < 2 \end{cases} \quad \text{and } g(x) = \overset{top}{x^3 - x^2 + 1} \text{ and } h(x) = \overset{bottom}{x^3 - 6x + 6}$$

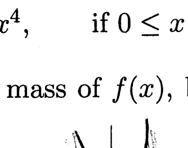
$x^3 - x^2 + 1 = x^3 - 6x + 6$

pts of intersection: $0 = x^2 - 6x + 5$ ⇒ $x = 1, 5$

Find (a) the center of mass of $f(x)$, bounded by $y = 0$ (b) the center of mass of the region bounded by $g(x)$ and $h(x)$.

(a)

f(x) =



$$m = \int_{-2}^{0} -3x\,dx + \int_0^2 x^4\,dy = \frac{62}{5}$$

$$M_y = \int_{-2}^{0} -3x^2 dx + \int_0^2 x^5\,dx = \frac{8}{3}$$

$$M_x = \frac{1}{2}\left[\int_{-2}^{0} 9x^2\,dx + \int_0^2 x^8\,dx\right] = \frac{364}{9}$$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{20}{93}, \frac{910}{279}\right)}$$

(b)

$$m = \int_1^5 (-x^2 + 6x - 5)\,dx = \frac{32}{3}$$

$$M_y = \int_1^5 (-x^3 + 6x^2 - 5x)\,dx = 32$$

$$M_x = \frac{1}{2}\int_1^5 \left[(x^3 - x^2 + 1)^2 - (x^3 - 6x + 6)^2\right]dx = \frac{3808}{15}$$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{119}{5}\right)}$$

10. Consider the differential equation

$$\frac{dy}{dt} = -y^3 + 11y^2 - 30y = -(y(y^2 - 11y + 30)) = -y(y-5)(y-6)$$

(a) What are the constant solutions? $\boxed{y=0,\ y=5,\ y=6}$

(b) When is y increasing? decreasing?

$y' = \frac{dy}{dt}$: $y'(-1) = 42$ $+$, $y'(1) = -20$ $-$, $y'(5.5) = 1.375$ $+$, $y'(7) = -14$ $-$

$y(t)$: $\nearrow$ 0 $\searrow$ 5 $\nearrow$ 6 $\searrow$

$\boxed{\begin{array}{l}\text{Incr: } (-\infty,0) \cup (5,6) \\ \text{Decr: } (0,5) \cup (6,\infty)\end{array}}$

(c) Determine all solutions to the differential equation when $y(1) = 2$.

$$\frac{dy}{y(y-5)(y-6)} = -dt$$

...partial fractions

$$-\frac{1}{5}\int \frac{dy}{y-5} + \frac{1}{6}\int \frac{dy}{y-6} + \frac{1}{30}\int \frac{dy}{y} = -\int dt$$

$$-\frac{1}{5}\ln|y-5| + \frac{1}{6}|y-6| + \frac{1}{30}\ln|y| = -t + C$$

$$\ln\left|\frac{y(y-6)^5}{(y-5)^6}\right| = -30t + C$$

$$\frac{y(y-6)^5}{(y-5)^6} = Ce^{-30t}$$ ...hmm...

$$\frac{2(-4)^5}{(-3)^6} = Ce^{-30} \Rightarrow C = \frac{-2048}{729}e^{30}$$

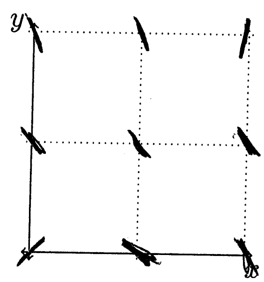
$$\Rightarrow \boxed{\frac{y(y-6)^5}{(y-5)^6} = \frac{-2048}{729}e^{30} \cdot e^{-30t}}$$

that's as far as you can go... SORRY!

11. Consider the differential equation

$$y' = x^2y^2 - 2x - 3y + 1$$

(a) Sketch the direction field for all integer pairs (x, y) shown on the graph below.



(b) Using a step size of 0.05, determine three approximate solutions of the differential equation starting at the point $(x, y) = (0, 1)$ $= y_3$

$\boxed{y_0 = 1 \qquad x_0 = 0}$

$y_1 = y_0 + h\, y'|_{(x_0, y_0)}$

$\boxed{y_1 = 1 + 0.05 \cdot (-2) = 0.9 \quad x_1 = 0 + 0.05 = 0.05}$

$y_2 = y_1 + h \cdot y'|_{(x_1, y_1)}$

$y_2 = 0.9 + 0.05 \cdot (-1.7979)$

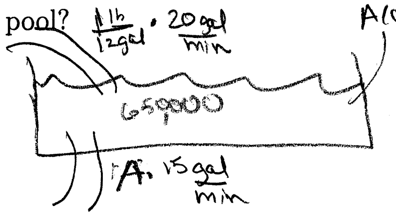
$\boxed{y_2 = 0.8101 \qquad x_2 = 0.10}$

$y_3 = 0.8101 + 0.05(-1.62374)$

$\boxed{y_3 = 0.728914 \qquad x_3 = 0.15}$

12. At the Natatorium in the U of M Rec Center, the pool there contains approximately 650,000 gallons of a water-chlorine mixture. Before admitting people to swim in the pool, the maintenance personnel ensure that there are 10,000 pounds of chlorine in the pool based on measurements and calculations. After allowing people to swim in the pool they add water-chlorine mixture, containing 1 pound of chlorine per 12 gallons, to the pool constantly at a rate of 20 gallons per minute. They also drain water off to prevent overflow at a rate of 15 gallons per minute. How much chlorine is in the pool one hour after admitting people into the pool?

$\frac{1\,lb}{12\,gal} \cdot \frac{20\,gal}{min}$ $\quad A(0)=10{,}000$



Let $A(t)$ denote the amt of chlorine at time t.

$$\frac{dA}{dt} = In - Out = \frac{1\,lb}{12\,gal}\cdot\frac{20\,gal}{min} - \frac{A}{650000+(20-15)t}\cdot\frac{15\,gal}{min}$$

(r_{in}, r_{out})

$$= \frac{5}{3} - \frac{3A}{130{,}000+t}$$

$$\frac{dA}{dt} + \frac{3}{130000+t}A = \frac{5}{3} \rightsquigarrow I.F. = e^{\int \frac{3\,dt}{130{,}000+t}} = (130{,}000+t)^3$$

$$\Rightarrow (130{,}000+t)^3\frac{dA}{dt} + 3(130{,}000+t)^2 A = \frac{5}{3}(130000+t)^3$$

$$\int \left((130{,}000+t)^3 A\right)' = \int \frac{5}{3}(130{,}000+t)^3$$

$$(130{,}000+t)^3 A = \frac{5}{12}(130{,}000+t)^4 + C$$

$$A = \frac{5}{12}(130{,}000+t) + C(130{,}000+t)^{-3}$$

Use I.C. $A(0) = 10{,}000$

$$10000 = \frac{5}{12}(130000) + \frac{C}{(130000)^3}$$

$$C = -\frac{132500}{3}\cdot(130{,}000)^3$$

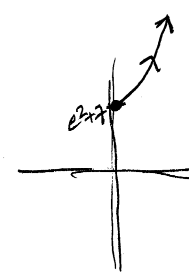
$$\Rightarrow \boxed{A(t) = \frac{5}{12}(130{,}000+t) - \frac{132500(130000)^3}{3(130{,}000+t)^3}}$$

13. Determine the Cartesian equation associated to the Parametric curves. Include a sketch and indicate the direction in which the curve is being traversed (i.e. as the parameter increases, which way does the curve go?).

$$x = \sqrt{t-2},\ y = e^t + 7,\ t \geq 2$$

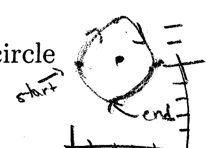
$x^2 = t-2$

$x^2+2 = t$ $\Rightarrow \boxed{y = e^{x^2+2} + 7}$



14. Determine the parametric equations for a particle that follows the circle

$$(x+3)^2 + (y-5)^2 = 4$$



and starts at $(-5,5)$, traverses clockwise, and ends at $(-3,3)$.

$x = 2\cos t$
$y = 2\sin t$ } counter clockwise!

with center $\Rightarrow$ $x = 2\cos t - 3$
$y = 2\sin t + 5$

For clockwise,

$x = 2\cos(-t) - 3$
$y = 2\sin(-t) + 5$ $\Rightarrow$ $\boxed{\begin{array}{l} x = 2\cos t - 3 \\ y = -2\sin t + 5 \end{array} \quad -\pi \leq t \leq \frac{\pi}{2}}$

15. In a forest, the fox and rabbit population can be modeled by the equations

$$\frac{dx}{dt} = 2x - 0.01xy \quad \rightarrow \boxed{\text{rabbits}}$$

$$\frac{dy}{dt} = -0.8y + 0.0002xy \quad \boxed{\text{foxes}}$$

(a) Which variable represents foxes? rabbits? why?

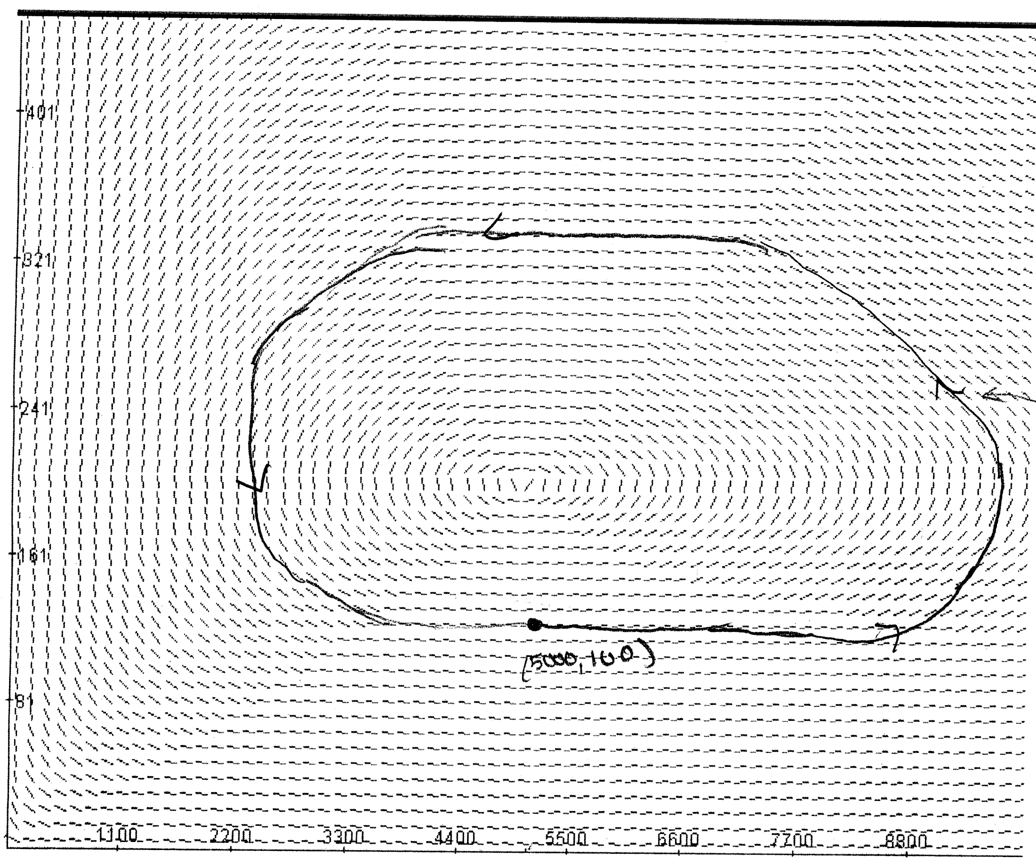
(b) What are the equilibrium solutions? $0 = x(2 - 0.01y)$ $\boxed{y = 200}$

$0 = 0.0002y(4000 - x)$ $\boxed{x = 4000}$

(c) Determine an expression for $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-0.8y + 0.0002xy}{2x - 0.01xy}}$$

(d) Sketch a phase portrait when there are 100 foxes and 5000 rabbits on the direction field below.



orientation

$\Downarrow$

$\frac{dx}{dt} = 2(5000) - 0.01(5000)(100)$

$= 5000 > 0$

$\Rightarrow$ counter clockwise

(e) Describe what happens to the fox and rabbit populations over time.

I'm lazy... Typically there isn't a lot of difficulty with this...

16. Consider the parameterization

$$x = -3t - t^3 \text{ and } y = 3t^2$$

(a) Determine all points where the tangent line has slope 1.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{-3-3t^2} = \frac{2t}{-1-t^2} = 1 \Rightarrow 2t = -1 - t^2$$

$$t^2 + 2t + 1 = 0$$

$$(t+1)^2 = 1 \Rightarrow t = -1$$

$$\Rightarrow \boxed{(x, y) = (4, 3)}$$

(b) Determine the length of the curve from $t = 3$ to $t = 8$.

$$L = \int_3^8 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_3^8 \sqrt{9t^4 + 18t^2 + 9 + 36t^2}\, dt = 3\int_3^8 \sqrt{t^4 + 6t^2 + 1}\, dt$$

$= \boxed{527.838}$ (try completing the square & doing a trig sub)

17. A cissoid of Diocles is given by the equation

$$r = \sin(\theta)\tan(\theta) = \frac{\sin^2\theta}{\cos\theta}$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

(a) Determine the Cartesian equation for this curve.

$$r\cos\theta = \sin^2\theta$$

$$x = \frac{y^2}{r^2}$$

$$x^3 + xy^2 = y^2$$

$$x^3 = y^2(1-x)$$

$$y^2(x-1) = x^2$$

$$\boxed{y^2 = \frac{x^3}{1-x}}$$

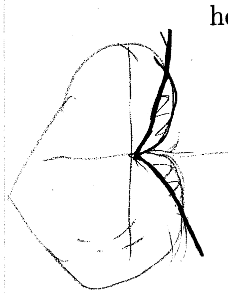
$$y = \pm \frac{x^{3/2}}{\sqrt{1-x}}$$

(b) Determine at what values of x there are vertical tangents.

$$2y\frac{dy}{dx} = \frac{(1-x)3x^2 + x^3}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{3x^2 - 2x^3}{2y(1-x)^2} = \frac{3x^2 - 2x^3}{2(1-x)^2} \cdot \frac{\sqrt{1-x}}{x^{3/2}} \Rightarrow \boxed{\text{VT when } x = 0, 1}$$

(c) Compute the Polar area between the cissiod and the cardioid $r = 4(1 - \cos(\theta))$. NOTE: the numbers here aren't pretty, but they are manageable...



$$A = 2 \cdot \frac{1}{2}\int_0^{\cos^{-1}(\frac{1}{3})} \left[(4(1-\cos\theta))^2 - (\sin\theta\tan\theta)^2\right] d\theta$$

Show your work, unlike me...

$$\approx \boxed{0.74817}$$

$$4 - 4\cos\theta = \frac{\sin^2\theta}{\cos\theta}$$

$$4\cos\theta - 4\cos^2\theta = \sin^2\theta = 1 - \cos^2\theta$$

$$0 = 3\cos^2\theta - 4\cos\theta + 1$$

$$\cos\theta = \frac{4 \pm \sqrt{16-12}}{6} = \frac{4 \pm 2}{6} = 1, \frac{1}{3}$$

$$\theta = 0, \cos^{-1}\left(\tfrac{1}{3}\right) \approx 70.5^\circ$$

Polar Roses!

18. Determine the area inside the curve $r_1 = \sin(3\theta)$ and outside the curve $r_2 = \cos(3\theta)$

each area is symmetric

$$A = 3\cdot\frac{1}{2}\left[\int_{\frac{\pi}{12}}^{\pi/6} (\cos(3\theta))^2 + \int_0^{\pi/12} (\sin(3\theta))^2 d\theta\right]$$

$\cos 3\theta = \sin 3\theta$

$1 = \tan 3\theta$

$\frac{\pi}{4} + \pi k = 3\theta$

$\theta = \frac{\pi}{12} + \frac{\pi}{3}\theta$

... show your work

$$= \boxed{\frac{\pi - 2}{8}}$$

* the reason this is $\pi/6$ is because a polar rose makes a complete cycle in π and each half petal contains $\pi/6$ radians. With the parametrization, we start at $(r,\theta) = (1,0)$ and end at $(0, \frac{\pi}{6})$ for the first half petal.

19. Consider the sequences

$$a_n = \frac{\cos^2(n)}{2^n}, \quad b_n = \frac{(-1)^{n-1}n^3}{n^3 + 2n^2 + 1}, \quad c_n = \frac{(\ln(n))^2}{n}$$

(a) Do the above sequences have limits? If so, find them; if not, show divergence.

a_n - Yes use Squeeze Thm to show $a_n \xrightarrow{n\to\infty} 0$

b_n - No, alternates b/w -1 & 1 for $n >> 0$

c_n - Yes! use L'H. $\frac{(\ln n)^2}{n} \to \frac{2\ln n}{n} \to 2\cdot\frac{1}{n} \to \boxed{0}$

(b) Are the sequences bounded? Show your work using the textbook's definition of bounded. Let $n > 0$

$a_n \to$ Yes $\to$ $\frac{0}{2^n} \le \frac{\cos^2}{2^n} \le \frac{1}{2^n}$ & the largest $\frac{1}{2^n}$ can be is 1, so $0 \le a_n \le 1$

$b_n \to$ Yes $\to$ for all $\forall n$, we can say certainly $-100 \le b_n \le 100$ as after a long time, $-1 \le b_n \le 1$ but initially it might not be in this range.

$c_n \to$ Hmm... depends! if $n = 0$ this is not bounded! if $n > 0$ Well, $\ln n < n \Rightarrow \frac{\ln n}{n} < 1$ $\xrightarrow[\text{value of } n]{\text{after some}}$ $\frac{(\ln n)^2}{n} < 1$ and $\frac{\ln n}{n} \ge 0 \ \forall n > 0$ so it is bounded, say by $-10 < c_n < 20$ to overshoot.

(c) Determine which sequences are monotonically increasing, monotonically decreasing, or neither.

a_n is monotonically decreasing (show by derivative)

b_n is neither (alternates)

c_n is monotonically decreasing for $n > e^2$ (show by the derivative)

20. Prove convergence or divergence of the series.

(a) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ Int test $\iff$ $\int_2^\infty \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x$, $du = \frac{1}{x}dx$

all ≥ 0

$= \int_{\ln 2}^{\infty} \frac{du}{\sqrt{u}} = 2u^{1/2}\Big|_{\ln 2}^{\infty} \to$ diverges

$\therefore \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ is divergent by the integral test.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-11}}{n^3+2n^2+5} < \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which conv. by p test with $p = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt{n^2-11}}{n^3+2n^2+5}$ converges by the Comp. Test.

(c) $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k} = \sum_{k=1}^{\infty} a_k$

$\sum_{k=1}^{\infty} \frac{5^k}{4^k} = \sum_{k=1}^{\infty} b_k$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\cancel{5^k}}{3^k+4^k} \cdot \frac{4^k}{5^k} \overset{\text{for } k>>0}{=} 1 > 0$

then, $\sum b_k$ div as it is a geometric series with $|r| > 1$

so $\sum_{k=1}^{\infty} a_k$ diverges by the limit comparison test.

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k\sqrt{k}}{k+5}$ $\rightsquigarrow$ use AST!

$a_k = \frac{\sqrt{k}}{k+5}$

$a_k \geq 0$ trivially

$\lim_{k\to\infty} a_k = 0$ trivially

$a_k' = \frac{(k+5)\frac{1}{2}k^{-1/2} - k^{1/2}\cdot 1}{(k+5)^2} \overset{?}{\leq} 0$

$\frac{k+5}{2k^{1/2}} \overset{?}{<} k^{1/2}$

$k+5 < 2k$

$5 < k$ $\rightsquigarrow$ so a_k is $\downarrow$ when $k>5$ ✓

$\therefore \sum_{k=1}^{\infty} (-1)^k a_k$ is conv by the AST

SHOW $\sum \frac{(-1)^k\sqrt{k}}{k+5}$ is NOT absolutely conv

$\Rightarrow$ $\boxed{\text{Conditionally Conv;}}$

(e) $\sum_{k=1}^{\infty} \frac{(-2)^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k^2+3k+2}$

Ratio test

$\lim_{k\to\infty} \left| \frac{2\cdot 2^k}{(k+1)^2+3(k+1)+2} \cdot \frac{k^2+3k+2}{2^k} \right|$

$= 2\lim_{k\to\infty} \left| \frac{k^2+3k+2}{k^2+5k+6} \right| = 2\cdot 1 > 0$

$\therefore$ by the Ratio test,

$\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k^2+3k+2}$ is divergent.

21. Determine the radius of convergence (ROC) and the interval of convergence (IOC) for the following power series:

$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2 5^n}$ ROC: $\lim_{n\to\infty} \left| \frac{x \cdot x^n}{(n+1)^2 5 \cdot 5^n} \cdot \frac{n^2 5^n}{x^n} \right| = \frac{|x|}{5} \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = \frac{|x|}{5} < 1$

$\Rightarrow |x| < \boxed{5 = ROC}$

IOC: guaranteed convergence on $|x|<5 \iff -5<x<5$, test endpts!

$x=5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \xRightarrow{\text{SHOW}}$ conv. by AST

$x=-5 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv by ptest as $p=2>1$ $\therefore \boxed{IOC = [-5, 5]}$

$(b)\ \sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$ ROC: $\lim_{n\to\infty} \left| \frac{2 \cdot 2^n (x-3)(x-3)^n}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n (x-3)^n} \right| = 2|x-3| \lim_{n\to\infty} \left|\frac{\sqrt{n+3}}{\sqrt{n+4}}\right|$

$= 2|x-3| < 1 \iff |x-3| < \boxed{\frac{1}{2} = ROC}$

IOC: guaranteed convergence on $-\frac{1}{2} < x-3 < \frac{1}{2} \iff \frac{5}{2} < x < \frac{7}{2}$ → look @ endpts!

$x=\frac{5}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n (-\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}} \xRightarrow{\text{SHOW}}$ conv. by AST

$x=\frac{7}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n (\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \xRightarrow{\text{SHOW}}$ div by lim comp test $\Rightarrow \boxed{IOC = [\frac{5}{2}, \frac{7}{2})}$

22. Find a power series representation of the functions

$(a) f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

$= \int \sum_{n=0}^{\infty} (-1)^n x^n dx + \int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$

Note $C=0$ as $\ln(\frac{1+0}{1-0}) = C$, $0 = C$

$= \sum_{n=0}^{\infty} \frac{[(-1)^n + 1] x^{n+1}}{n+1}$ Let $m = n+1$

$\Rightarrow \boxed{\sum_{m=1}^{\infty} \frac{[(-1)^{m-1}+1] x^m}{m} = \sum_{m=1}^{\infty} \frac{2x^{2m-1}}{2m-1}}$

$(b) f(x) = \frac{4}{x^2 - 12x + 32}$ Partial fractions $\frac{1}{x-8} - \frac{1}{x-4} = -\frac{1}{8} \cdot \frac{1}{1-\frac{x}{8}} + \frac{1}{4} \cdot \frac{1}{1-\frac{x}{4}}$

$= -\frac{1}{8} \sum_{n=0}^{\infty} \frac{x^n}{8^n} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{x^n}{4^n} = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}} - \sum_{n=0}^{\infty} \frac{x^n}{8^{n+1}} = \sum_{n=0}^{\infty} \frac{(2^{n+1}-1)x^n}{8^{n+1}}}$ $= \frac{1}{8} + \frac{3}{64}x + \frac{7}{512}x^2 + \frac{15}{4096}x^3$

23. Using power series, approximate the definite integral to six decimal places

$$\int_0^{0.5} 4x\sin(2x^3)\,dx$$

$\sin(2x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{6n+3}}{(2n+1)!}$

$4x\sin(2x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+3} x^{6n+4}}{(2n+1)!}$

$\int_0^{0.5} 4x\sin(2x^3)dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+3} (0.5)^{6n+5}}{(6n+5)(2n+1)!}$

Use Ast Estimation Thm

$|R_N| = \frac{2^{2N+5}(0.5)^{6N+11}}{(6N+11)(2N+3)!} < 1\times10^{-6}$

By calculation,

$N=1$ works

but $N=n+1 \Rightarrow N=0$

so $\frac{(-1)^0 2^3 (0.5)^5}{5\cdot 1!} = \boxed{0.05}$ is an accurate sum to 6 decimal places

24. Determine the Taylor series for $f(x) = 1/\sqrt{x}$ centered around $x = 4$.

$f(x) = \frac{1}{\sqrt{(x-4)+4}} = ((x-4)+4)^{-1/2} = \frac{1}{2}\left(\left(\frac{x-4}{4}\right)+1\right)^{-1/2}$

$= \boxed{\frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x-4}{4}\right)^n = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{(x-4)^n}{2^{2n+1}}}$

25. Write out the first three nonzero terms in the Maclaurin series for the function $f(x) = e^x \arctan(x)$

$f(x) = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}\pm\cdots\right)$

$f(x) = (0)1 + (1)x + (1)x^2 + \left(-\frac{1}{3}+\frac{1}{2}\right)x^3 + \left(-\frac{1}{3}+\frac{1}{6}\right)x^4 + \cdots$

$\boxed{f(x) = x + x^2 + \frac{x^3}{6} + \cdots}$

26. Find the sum of the series

(a) $\sum_{n=0}^{\infty} \frac{1}{n^2+9x+18} = \sum_{n=0}^{\infty} \frac{1}{(n+6)(n+3)} \overset{\text{partial fractions}}{=} \frac{1}{3}\left[\sum_{n=0}^{\infty}\frac{1}{n+3} - \sum_{n=0}^{\infty}\frac{1}{n+6}\right]$ write it out!

$= \frac{1}{3}\left[\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots\right) - \left(\frac{1}{6}+\frac{1}{7}+\cdots\right)\right] = \frac{1}{3}\left[\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right] = \boxed{\frac{47}{180}}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n e^{4n}}{729^n (2n)!} \quad \sum_{n=0}^{\infty} \frac{(-1)^n (e^2)^{2n}}{(3^3)^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{e^2}{27}\right)^{2n}}{(2n)!} = \boxed{\cos\left(\frac{e^2}{27}\right)}$

27. Let

$$\vec{r} = 3\hat{\imath} + 5\hat{\jmath} - 6\hat{k},\ \vec{s} = -4\hat{\imath} + \hat{\jmath} + 2\hat{k},\ \vec{t} = 6\hat{\imath} + 14\hat{\jmath} + 5\hat{k}$$

(a) Compute the angle formed by $\vec{r}$ and $\vec{s}$ as well as the angle formed by $\vec{s}$ and $\vec{t}$. If the angle is $\pi/2$, what does that mean about the vectors?

$$\theta = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{s}}{\|\vec{r}\|\,\|\vec{s}\|}\right) = \cos^{-1}\left(\frac{-12+5-12}{\sqrt{70}\sqrt{21}}\right) = \cos^{-1}\left(\frac{19}{\sqrt{1470}}\right) \approx 1.0523 \text{ rad} \approx 60.24^\circ$$

$$\theta = \cos^{-1}\left(\frac{\vec{s}\cdot\vec{t}}{\|\vec{s}\|\,\|\vec{t}\|}\right) = \cos^{-1}\left(\frac{-24+14+10}{\sqrt{21}\cdot\sqrt{257}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

$\vec{s}$ + $\vec{t}$ are orthogonal (perpendicular, $\perp$)

(b) Find a vector that is perpendicular to both $\vec{r}$ and $\vec{t}$.

determinant!

cofactor method

$$\vec{r}\times\vec{t} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & -6 \\ 6 & 14 & 5 \end{vmatrix} = \begin{vmatrix} 5 & -6 \\ 14 & 5 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 3 & -6 \\ 6 & 5 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 3 & 5 \\ 6 & 14 \end{vmatrix}\hat{k}$$

$$= (25+84)\hat{\imath} - (15+36)\hat{\jmath} + (42-30)\hat{k}$$

$$= \boxed{109\hat{\imath} - 51\hat{\jmath} + 12\hat{k} = \langle 109, -51, 12\rangle}$$

YAY MATH!